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Stress tensor conformal anomaly for Weinberg-type fields in curved spacetimes

N D Birrell

Department of Mathematics, King's College, Strand, London WC2R 2LS, UK

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Abstract. The coefficient of the $R_{\mu\nu}R^{\mu\nu}-\frac{1}{3}R^2$ part of the conformal anomaly of the stress tensor for arbitrary spin Weinberg-type fields in a curved background spacetime is calculated. The method used involves obtaining terms in the DeWitt–Schwinger expansion for the two-point function by explicit calculation of the complete two-point function in de Sitter space.

1. Introduction

Since the realisation that the quantum expectation value of operators which are classically tracefree, such as the conformal stress tensor, may develop a trace when calculated in a curved background spacetime (Capper and Duff 1974, Davies *et al* 1976), and that this can be a quite generally occurring phenomenon (Deser *et al* 1976), there has been considerable activity in calculating such anomalies for various different fields by a variety of methods (Duncan 1977, Brown 1977, Bunch and Davies 1977, Dowker 1977, 1978, Dowker and Critchley 1977, Duff 1977, Tsao 1977, Christensen and Duff 1978).

The occurrence of these anomalies has consequences in many areas, among the most important of these being the relation between the stress tensor conformal anomaly and the Hawking flux from a black hole (Christensen and Fulling 1977, Birrell and Davies 1978). More recently the importance of anomalies to supersymmetric and supergravity theories has been stressed (Christensen and Duff 1978). This latter paper also presents the results of the first calculation of the axial current and conformal trace anomalies for arbitrary spin quantum fields. Actually only a part of the complete trace anomaly is calculated in the work of Christensen and Duff, namely the $C_{\mu\nu\omega\rho}C^{\mu\nu\omega\rho}$ part, where C is the Weyl tensor.

It has been shown (Deser *et al* 1976) that the most general form for the anomaly in the renormalised stress tensor expectation value is

$$\langle T^{\mu}_{\mu}\rangle=-k_1C_{\mu\nu\rho\omega}C^{\mu\nu\rho\omega}-k_2(R_{\mu\nu}R^{\mu\nu}-\tfrac{1}{3}R^2)+k_3\Box R-k_4R^2 \tag{1.1}$$

(sign conventions are as in Davies *et al* 1977). Subsequently Duff (1977) has shown that the so-called anomaly coefficients must satisfy

$$k_4=0 \tag{1.2a}$$

$$2k_1+k_2-3k_3=0.† \tag{1.2b}$$

† Equation (1.2*b*) requires some qualification which we shall give in § 5.

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In this paper we complete the determination of the anomaly for higher spin fields of type $(s, 0)$ by calculating the coefficient k_2 in (1.1). The remaining coefficients are then found using the result of Christensen and Duff (1978) and equations (1.2). The value of k_2 is found by calculating it directly in de Sitter space using the results of the work of Grensing (1977). Due to this calculational method the class of higher spin fields for which k_2 is obtained is restricted to that of the type discussed at length by Weinberg (Weinberg 1964a, b, c; see also the work of Dowker and Dowker 1966a, b, Dowker 1967, and the review of Mohan 1968). These fields constitute a subset of the general fields used by Christensen and Duff in the calculation of k_1, and the results of this paper would provide a check on their method if it were applied to a calculation of the k_2 coefficient.

In the next section we discuss some of the properties of Weinberg-type fields in curved spacetimes, in particular giving consideration to the question of the conformal invariance of their field equations, while in § 3 we specialise to de Sitter space to review and correct the calculations of Grensing (1977). In § 4 we present the calculation of k_2 using the point separation method as the regularisation scheme (see e.g. Bunch and Davies 1978). The final section summarises the formulae for all the anomaly coefficients and compares the results with those of other authors for specific spins. We also mention that the complete stress tensor for arbitrary Robertson–Walker spacetimes is determined by these results.

2. Weinberg-type fields in curved spacetimes

We firstly briefly review the subject of Weinberg-type fields in flat space. The notation that we use is mainly that of Weinberg (Weinberg 1972, especially §§ 2.12 and 12.5 which contain an excellent introduction to the matters relevant to this work).

The spin of a quantum field is determined by the representation of the Lorentz group under which it transforms. That is, for a Lorentz transformation $\Lambda^{\alpha}_{\beta}(x)$ a field ψ_n (n labels the field components) will transform as

$$\psi_n(x) \to \sum_m [D(\Lambda(x))]_{nm}\psi_m(x) \tag{2.1}$$

where $D(\Lambda)$ is some matrix representation of the Lorentz group. For an infinitesimal Lorentz transformation

$$\Lambda^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} + \omega^{\alpha}_{\beta}$$

we have

$$D(1+\omega) = 1 + \tfrac{1}{2}\omega^{\alpha\beta}\sigma_{\alpha\beta} \tag{2.2}$$

where $\sigma_{\alpha\beta}$ are the antisymmetric generators satisfying

$$[\sigma_{\alpha\beta}, \sigma_{\gamma\delta}] = \eta_{\gamma\beta}\sigma_{\alpha\delta} - \eta_{\gamma\alpha}\sigma_{\beta\delta} + \eta_{\delta\beta}\sigma_{\gamma\alpha} - \eta_{\delta\alpha}\sigma_{\gamma\beta} \tag{2.3}$$

$\eta_{\alpha\beta}$ being the Minkowski metric $(+---)$. A representation of the Lorentz group is specified by giving $\sigma_{\alpha\beta}$.

The Weinberg-type fields used in this paper are defined to transform under the $(s, 0)\oplus(0, s)$ representation of the Lorentz group:

$$\sigma_{i0} = \begin{cases} -J_i & (s, 0) \\ +J_i & (0, s) \end{cases} \tag{2.4}$$

$$\sigma_{ij} = -\mathrm{i}\epsilon_{ijk}J_k.\dagger \tag{2.5}$$

Here the J_i are the generators of the usual $(2s+1)$-dimensional representation of the rotation group:

$$[J_1^{(s)} \pm \mathrm{i}J_2^{(s)}]_{nm} = \delta_{nm\pm1}[(s\mp m)(s\pm m+1)]^{1/2} \tag{2.6}$$

$$[J_3^{(s)}]_{nm} = \delta_{nm}m. \tag{2.7}$$

These $2(2s+1)$-component fields are constructed explicitly by Weinberg. In particular (Weinberg 1964b) he writes the massive fields in a helicity basis, a method which we use in de Sitter space in § 3. The fields of mass m obey the Klein–Gordon equation

$$\eta^{\alpha\beta}\,\partial_\alpha\,\partial_\beta\psi_n + m^2\psi_n = 0 \tag{2.8}$$

and, because ψ has twice as many components as are needed, the $(s, 0)$ and $(0, s)$ components of the field obey equations linking the two. We do not need these fairly complicated equations in the general case, but note that for $m = 0$ they decouple and may be written in one equation as (e.g. Wichmann 1962; see also Appendix 1)

$$\sigma^{\alpha\beta}\,\partial_\beta\psi + s\eta^{\alpha\beta}\,\partial_\beta\psi = 0. \tag{2.9}$$

The $\alpha = 0$ component gives equations (4.19) and (4.20) of Weinberg (1964b).

Let us now turn to the treatment of such fields in a curved background spacetime with metric $g_{\mu\nu}$. Following Weinberg (1972, § 12.5) we write the metric at a point x in terms of a tetrad (*vierbein*) there:

$$g_{\mu\nu}(x) = V_\mu^\alpha(x)V_\nu^\beta(x)\eta_{\alpha\beta}. \tag{2.10}$$

We use letters from the beginning of the Greek alphabet as tetrad labels, which are raised and lowered by the flat spacetime metric. (We also have $g^{\mu\nu} = V_\alpha^\mu V_\beta^\nu\eta^{\alpha\beta}$, $V_\alpha^\mu V_\mu^\beta = \delta_\alpha^\beta$, $V_\alpha^\mu V_\nu^\alpha = \delta_\nu^\mu$.) A covariant derivative $\mathscr{D}_\alpha$ is defined by

$$\mathscr{D}_\alpha = V_\alpha^\mu(\partial_\mu + \Gamma_\mu) \tag{2.11}$$

where the spin connection Γ_μ is given by

$$\Gamma_\mu = \tfrac{1}{2}\sigma^{\alpha\beta}V_\alpha^\nu V_{\beta\nu;\mu} \tag{2.12}$$

the $\sigma^{\alpha\beta}$ being the generators of the representation of the Lorentz group under which the object being differentiated transforms. Weinberg then shows that the effect of gravitation (i.e. a non-Minkowski metric) on a physical system is taken into account by replacing all derivatives ∂_α in the field equations or actions of special relativity by the covariant derivative $\mathscr{D}_\alpha$. Thus the field equation for Weinberg-type fields in curved space is simply given by (2.8) with such a replacement.

Let us consider the half of

$$\psi = \begin{pmatrix} \psi^{(s,0)} \\ \psi^{(0,s)} \end{pmatrix}$$

which transforms under the $(s, 0)$ representation of the Lorentz group (the treatment of $\psi^{(0,s)}$ is identical, or indeed they can be treated together with little extra difficulty). When we differentiate $\psi^{(s,0)}$ the first time to form $\mathscr{D}_\beta\psi^{(s,0)}$ the spin connection (2.12) will be formed with the generators of the $(s, 0)$ representation of the Lorentz group

† Note that the overall minus sign difference between (2.4) and (2.5) and the corresponding equations in Weinberg (1964a, b, c) is due to the difference in the signature of the metric used.

(2.4)–(2.6). Then $\mathscr{D}_\beta\psi^{(s,0)}$ transforms under the $(\frac{1}{2},\frac{1}{2})\oplus(s,0)$ representation, and so when it is differentiated the generators of this representation must be used in constructing the spin connection. If we denote the generators and spin connection for the $(\frac{1}{2},\frac{1}{2})\oplus(s,0)$ representation by $[\sigma_{\alpha\beta}]^{\gamma n}_{\delta m}$ and $[\Gamma_\mu]^{\gamma n}_{\delta m}$ respectively, then it is a straightforward task to write them in terms of the corresponding $(\frac{1}{2},\frac{1}{2})$ and $(s,0)$ quantities as

$$[\sigma_{\alpha\beta}]^{\gamma n}_{\delta m}=\delta^n_m[\sigma_{\alpha\beta}]^\gamma_\delta+\delta^\gamma_\delta[\sigma_{\alpha\beta}]^n_m \tag{2.13}$$

$$[\Gamma_\mu]^{\gamma n}_{\delta m}=\delta^n_m[\Gamma_\mu]^\gamma_\delta+\delta^\gamma_\delta[\Gamma_\mu]^n_m \tag{2.14}$$

where the type of label affixed to an object specifies to which representation it belongs (i.e. Greek indices indicate the vector representation while Latin indices denote the $(s,0)$ representation). For the $(\frac{1}{2},\frac{1}{2})$ vector representation the generators are given by

$$[\sigma_{\alpha\beta}]^\gamma_\delta=\delta^\gamma_\alpha\eta_{\beta\delta}-\delta^\gamma_\beta\eta_{\alpha\delta}. \tag{2.15}$$

Bearing these considerations in mind we can write the equation for Weinberg-type fields in curved space as

$$V^\mu_\alpha[\delta^\alpha_\gamma\delta^n_m\,\partial_\mu+[\Gamma_\mu]^{\alpha n}_{\gamma m}]V^\gamma_\nu[\delta^m_p\,\partial^\nu+[\Gamma^\nu]^m_p]\psi^p+m^2\psi^n=0. \tag{2.16}$$

For clarity we have shown all the indices in (2.16), but shall henceforth use a highly abbreviated notation to denote this equation by

$$(\nabla^2+m^2)\psi=0. \tag{2.16$'$}$$

(For a Lagrangian giving these field equations for $m=0$ see Dowker and Dowker (1966a).)

Now if we are to calculate a conformal anomaly it is imperative that the massless field equation be invariant under conformal transformations of the metric

$$g_{\mu\nu}\to\bar{g}_{\mu\nu}=\Omega^2g_{\mu\nu} \tag{2.17}$$

where these are accompanied by transformations

$$\psi(x)\to\bar{\psi}(x)=\Omega^w(x)\psi(x) \tag{2.18}$$

of the field (w being known as the conformal weight of the field). This symmetry then goes over to the massless stress tensor as well, making it tracefree (see e.g. Brown and Cassidy 1977, Christensen 1978). Thus the only contribution to the trace comes from the massive part, giving for its expectation value

$$\langle T^\mu_\mu\rangle=m^2\lim_{x''\to x'}\mathrm{Tr}\langle 0|\psi(x'')\bar{\psi}(x')|0\rangle \tag{2.19}$$

where $\bar{\psi}=(\psi^{(0,s)\dagger},\psi^{(s,0)\dagger})$. In the limit as the points come together, this quantity is infinite and needs to be renormalised. The conformal trace anomaly develops in the renormalisation process, which results in the trace of the renormalised massless stress tensor being non-zero. We shall return to this point in § 4.

It is thus clear that we must check on the conformal invariance of (2.16) for $m=0$. After some algebra (see Appendix 1) we obtain $\bar{\nabla}\bar{\psi}$, the left-hand side of the conformally transformed massless field equation, in terms of untransformed quantities to be

$$\begin{aligned}\bar{\nabla}^2\bar{\psi}=\Omega^{w-2}\nabla^2\psi+(2+2w)\Omega^{w-3}\Omega_{,\nu}(\partial^\nu+\Gamma^\nu)\psi+[w^2+w-s(s+1)]\Omega_{,\mu}\Omega^{,\mu}\Omega^{w-4}\psi\\-2\Omega^{w-4}\Omega_{,\mu}V^\mu_\alpha V^\nu_\beta\sigma^{\alpha\beta}(\partial_\nu+\Gamma_\nu)\psi+w\Omega^{w-3}\Omega^{;\mu}_{;\mu}\psi.\end{aligned} \tag{2.20}$$

At first sight this seems to offer very little chance of giving conformal invariance, for which we should need everything but the first term on the right-hand side of (2.20) to vanish. The situation is saved, however, by noting that in flat space the fields obey the extra conditions (2.9). If we apply the same procedure as was used in taking equation (2.8) to curved space we find the curved space equivalent of (2.9) to be

$$V^{\mu}_{\alpha} V^{\nu}_{\beta} \sigma^{\alpha\beta} (\partial_{\nu} + \Gamma_{\nu})\psi + s g^{\mu\nu} (\partial_{\nu} + \Gamma_{\nu})\psi = 0. \qquad (2.21)$$

Further, one can easily verify (see Appendix 1) that this equation is conformally invariant if the field is given weight $w = -s - 1$. Thus, as the fields satisfying the massless version of the flat equation (2.8) also satisfy (2.9) (Weinberg 1964b, or see Appendix 1), then in a conformally flat spacetime (e.g. de Sitter space) the solutions of the conformally invariant (massless, curved space) equivalent of (2.8) will also satisfy (2.21).

If we make use of (2.21) we can reduce (2.20) to

$$\bar{\nabla}^2 \bar{\psi} = \Omega^{w-2} \nabla^2 \psi + (2 + 2w + 2s) \Omega^{w-2} \Omega_{,\nu} (\partial^{\nu} + \Gamma^{\nu}) \psi$$
$$+ [w^2 + w - s(s+1)] \Omega^{w-4} \Omega_{,\mu} \Omega^{,\mu} \psi + w \Omega^{w-3} \Omega^{;\mu}_{;\mu} \psi.$$

If we now take $w = -s - 1$ we obtain

$$\bar{\nabla}^2 \bar{\psi} = \Omega^{-s-3} \nabla^2 \psi - (s+1) \Omega^{-s-4} \Omega^{;\mu}_{;\mu} \psi. \qquad (2.22)$$

Noting that under the conformal transformation (2.17) the Ricci curvature scalar in four dimensions transforms as

$$R \rightarrow \bar{R} = \Omega^{-2} R + 6 \Omega^{-3} \Omega^{;\mu}_{;\mu} \qquad (2.23)$$

we see that if instead of equation (2.16) we adopt the equation

$$(\nabla^2 + m^2 + \xi R) \psi = 0 \qquad (2.24)$$

with $\xi = (s+1)/6$ (henceforth called conformally coupled), then conformal invariance for $m = 0$ will result.

In the next section we shall verify explicitly that in the massless limit the de Sitter space field for $\xi = (s+1)/6$ is related to the flat space fields of Weinberg by a conformal transformation, as predicted by the analysis above.

3. Weinberg-type fields in de Sitter space

Grensing (1977) has considered the solution of an equation similar to (2.24) in de Sitter space. As the results obtained here disagree with Grensing's on one crucial point we shall repeat a few of the steps leading to the fields.

The metric for de Sitter space is

$$g_{\mu\nu} = (r^2/\eta^2) \eta_{\mu\nu} \qquad (3.1)$$

where η is the conformal time parameter and r is a constant related to the Ricci scalar by

$$R = 12/r^2. \qquad (3.2)$$

The form of (2.24) in de Sitter space is most easily obtained by writing it in terms of the flat space quantities $V^{\mu}_{\alpha} = \delta^{\mu}_{\alpha}$, $\Gamma_{\mu} = 0$ using (A1.4) with $\Omega = r/\eta$; this gives

$$[\eta^2 \partial_{\alpha} \partial^{\alpha} \psi - 2\eta (\partial/\partial\eta) + 2\eta \sigma^{0\alpha} \partial_{\alpha} + 12\xi + m^2 r^2 - s(s+1)] \psi = 0. \qquad (3.3)$$

Following Grensing we write the positive and negative frequency parts of $\psi^{(s,0)}$ as

$$\chi^{\pm}(\eta, \boldsymbol{x}) = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^{3/2}} \exp(\pm \mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x})\chi^{\pm}(\eta, \boldsymbol{p}). \tag{3.4}$$

Equation (3.3) then gives

$$\left[\eta^2 \frac{\mathrm{d}^2}{\mathrm{d}\eta^2} - 2\eta\frac{\mathrm{d}}{\mathrm{d}\eta} + (\eta p)^2 \mp 2\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{J} + 12\xi + m^2r^2 - s(s+1)\right]\chi^{\pm}(\eta, \boldsymbol{p}) = 0 \tag{3.5}$$

where we have used (2.4). Writing $\chi^{\pm}(\eta, \boldsymbol{p})$ in a helicity basis

$$\begin{aligned}\chi^{+}(\eta, \boldsymbol{p}) &= \sum_{s_3} a^{+}(\eta, \boldsymbol{p}, s_3)\chi(\hat{p}, s_3)\\ \chi^{-}(\eta, \boldsymbol{p}) &= \sum_{s_3} c^{-}(\eta, \boldsymbol{p}, s_3)\chi(\hat{p}, s_3)\end{aligned} \tag{3.6}$$

where

$$\chi_n(\hat{p}, s_3) = D^{(s)}_{n,s_3}[R(\hat{p})] \tag{3.7}$$

and the rotation matrix $D^{(s)}$ is defined in Appendix 1, we find as the equation for a^{+}

$$\left[\eta^2 \frac{\mathrm{d}^2}{\mathrm{d}\eta^2} - 2\eta\frac{\mathrm{d}}{\mathrm{d}\eta} + (\eta p)^2 - 2\mathrm{i}p\eta s_3 + 12\xi + m^2r^2 - s(s+1)\right]a^{+}(\eta, \boldsymbol{p}, s_3) = 0. \tag{3.8}$$

In obtaining this result we have also made use of equation (A1.10) (with $-s$ replaced by s_3). Equation (3.8) is of Whittaker's type (Whittaker and Watson 1927) and we take the solution

$$a^{+}(\eta, \boldsymbol{p}, s_3) = a^{+}(\boldsymbol{p}, s_3)\frac{\eta}{2rp^{1/2}} \exp\left(\frac{\mathrm{i}\pi s_3}{2}\right)(mr)^{s_3} W_{-s_3,\nu}(2\mathrm{i}p\eta) \tag{3.9}$$

where $\nu^2 = \frac{9}{4} - 12(m^2R^{-1} + \xi) + s(s+1)$. The solution for c^{-} is found in a similar way to be

$$c^{-}(\eta, \boldsymbol{p}, s_3) = c^{-}(\boldsymbol{p}, s_3)\frac{\eta}{2rp^{1/2}} \exp\left(-\frac{\mathrm{i}\pi s_3}{2}\right)(mr)^{s_3} W_{-s_3,\nu}(-2\mathrm{i}p\eta). \tag{3.10}$$

The only difference which occurs in finding $\psi^{(0,s)}$ is due to the difference in (2.4) between the $(s, 0)$ and $(0, s)$ generators, and we obtain for the total field

$$\begin{aligned}\psi(\eta, x) = \sum_{s_3} \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^{3/2}}&[\exp(\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x})a^{+}(\boldsymbol{p}, s_3)\psi^{+}(\eta, \boldsymbol{p}; s_3)\\ &+\exp(-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x})c^{-}(\boldsymbol{p}, s_3)\psi^{-}(\eta, \boldsymbol{p}; s_3)]\end{aligned} \tag{3.11}$$

where

$$\psi^{+}(\eta, \boldsymbol{p}; s_3) \equiv \frac{\eta}{2rp^{1/2}}\begin{pmatrix} \exp\left(\frac{\mathrm{i}\pi s_3}{2}\right)(mr)^{s_3} W_{-s_3,\nu}(2\mathrm{i}p\eta)\chi(\hat{p}, s_3) \\ \exp\left(-\frac{\mathrm{i}\pi s_3}{2}\right)(mr)^{-s_3} W_{s_3,\nu}(2\mathrm{i}p\eta)\chi(\hat{p}, s_3)\end{pmatrix} \tag{3.12}$$

$$\psi^{-}(\eta, \boldsymbol{p}; s_3) \equiv \frac{\eta}{2rp^{1/2}} \begin{pmatrix} \exp\left(-\frac{\mathrm{i}\pi s_3}{2}\right)(mr)^{s_3} W_{-s_3,\nu}(-2\mathrm{i}p\eta)\chi(\hat{p}, s_3) \\ (-1)^{2s} \exp\left(-\frac{3\pi \mathrm{i} s_3}{2}\right)(mr)^{-s_3} W_{s_3,\nu}(-2\mathrm{i}p\eta)\chi(\hat{p}, s_3) \end{pmatrix}.$$

Apart from the normalisation which will be explained shortly, the obvious difference between Grensing's result and (3.12) is that he has used M-type Whittaker functions while (3.12) contains W-type functions. Grensing chooses M-type functions by consideration of the process in which $r \to \infty$ and de Sitter space goes over to Minkowski space, demanding that the modes be pure positive or negative frequency there. To achieve this analysis he uses the asymptotic expansions of the Whittaker functions M_{km} for large m given by Kazarinoff (1955). However, as Kazarinoff points out, these expressions only include dominant terms and thus do not give a true indication of whether or not the modes contain only positive or negative frequency components. It is clear for a number of reasons that it is in fact the W-type Whittaker functions which should be taken:

(i) In the $s = 0$ case the modes (3.12) reduce to Hankel function solutions used by other authors in studies of the scalar field in de Sitter space (e.g. Nachtmann 1967, Bunch and Davies 1978), and are generally accepted as defining the vacuum in de Sitter space. In particular Nachtmann (1967) shows that they do indeed contract to pure positive and negative frequency modes as $r \to \infty$. Moreover, Grensing himself uses Hankel function modes in studying the scalar field commutator at a later point in his paper. On the other hand, the M-type Whittaker functions reduce to I-type modified Bessel functions.

(ii) As $\eta \to \infty$ the rate of expansion of de Sitter space goes to zero and thus, from the work of Parker and Fulling (1974), we expect ψ^{+} to go over to a pure positive frequency mode. In this limit we have (Bucholz 1969, chap. III)

$$W_{-s_3,\nu}(2\mathrm{i}p\eta) \to (2\mathrm{i}p\eta)^{-s_3} \exp(-\mathrm{i}p\eta)$$

while

$$M_{s_3,\nu}(2\mathrm{i}p\eta) \to \Gamma(1+2\nu)\left(\frac{(2\mathrm{i}p\eta)^{-s_3}\exp(\mathrm{i}p\eta)}{\Gamma(\frac{1}{2}+\nu-s_3)} + \frac{(2\mathrm{i}p\eta)^{s_3}\exp[-\mathrm{i}\pi(s_3-\frac{1}{2}-\nu)]\exp(-\mathrm{i}p\eta)}{\Gamma(\frac{1}{2}+\nu+s_3)}\right). \tag{3.13}$$

Clearly the W Whittaker functions give purely positive frequencies, while the M functions give a mixture. Equation (3.13) also sheds some light on the situation which arises when Grensing considers $r \to \infty$, for then, with $\nu = -\mathrm{i}\rho$ as is the case in his $\psi^{+}_{(s,0)}$, the second, positive frequency, part of (3.13) is dominant, thus falsely giving the impression that the M-type functions can be used in the fields.

(iii) The final check that we have taken the correct Whittaker functions in (3.12) involves taking the massless limit. As was decided at the end of the last section, this should give the usual flat space modes (Weinberg 1964b) multiplied by a conformal factor, provided we take $\xi = (s+1)/6$ (conformal coupling). With this choice of ξ we have

$$\nu^2 = [(2s-1)/2]^2 - 12m^2R^{-1}. \tag{3.14}$$

Following Weinberg (1964b) we must multiply by m^s before taking the limit $m \to 0$.

Doing this for example in $\psi^{+}_{(s,0)}$, from (3.12) we obtain

$$m^s\psi^{+}_{(s,0)} \xrightarrow{m\to 0} \frac{\eta}{2rp^{1/2}}\delta_{s_3,-s}\exp\left(-\frac{\mathrm{i}\pi s}{2}\right)r^{-s}W_{s,s-1/2}(2\mathrm{i}p\eta)\chi(\hat{p},-s)$$

$$=\frac{\delta_{s_3,-s}}{\sqrt{2}}\left(\frac{\eta}{r}\right)^{s+1}(2p)^{s-1/2}\exp(-\mathrm{i}p\eta)\chi(\hat{p},-s)$$

the value of the Whittaker function being obtained from Bucholz (1969, Appendix I). Apart from a factor of $1/\sqrt{2}$ due to our use of $2(2s+1)$-component fields, and the difference due to the different signature metric, this when inserted in (3.11) gives precisely the expected conformal factor multiplying the equivalent flat space expression of Weinberg (1964b, equation (4.10)). With a normalisation proportional to $1/m$ forced upon one if M-type Whittaker functions are used (Grensing 1977), such a reduction is not possible in that case.

It only remains to mention the differences in normalisation of the field due to the use of W Whittaker functions. The fields are normalised by the same prescription as was used by Grensing, all the differences, except for the factors $(mr)^{\pm s_3}$, arising from differences in the Wronskians of the two types of Whittaker functions (Bucholz 1969, p 25). Because of the form of the normalisation condition one is able to insert $(mr)^{+s_3}$ in the top components of (3.12) provided a compensating factor of $(mr)^{-s_3}$ is included in the bottom components. This manipulation simply makes the massless limit obvious, but could of course be left out without affecting any of the results of the next section.

4. Calculation of the anomaly

As was pointed out in § 2, the conformal trace anomaly arises because of the necessity to renormalise the stress tensor, and in particular its trace (2.19). The way in which the anomaly arises using the various available techniques for initially regularising the stress tensor has been much discussed in the references already cited and will not be reviewed here. We should, however, mention that there is general agreement between the values of the k_2 anomaly coefficients obtained using the different methods.† We shall simply adopt the point-separation regularisation scheme as applied by Bunch and Davies (1978) (see also Bunch *et al* 1978), although it is fairly obvious that had we used the adiabatic regularisation scheme of Parker and Fulling (1974) as implemented by Birrell (1978) exactly the same result would be obtained.

The method advocated by Bunch and Davies, and described by them in some detail (Bunch and Davies 1978), involves expanding the point-separated stress tensor in powers of the separation distance and renormalising by subtracting from this terms up to adiabatic order four (order R^2/m^2) in the DeWitt–Schwinger expansion of the stress tensor (DeWitt 1965, 1975, Schwinger 1951). The anomaly arises from the term of order $1/m^2$ multiplied by the m^2 in equation (2.19).

Now the DeWitt–Schwinger expansion has been calculated explicitly by Christensen (1978) for spins 0, $\frac{1}{2}$ and spin 1 (Maxwell) fields, allowing these anomalies to be read off. However, lacking such results for the Weinberg fields under consideration, we can obtain the relevant terms in the expansion by following Bunch and Davies and

† Apart from the recent work of Brown and Dutton (1978), in which the stress tensor does not naturally acquire an anomaly—although one can be added if desired.

performing an asymptotic expansion in R/m of the two-point function in (2.19). For further discussion of this point readers are referred to Bunch and Davies (1978).

The evaluation of the integrals in forming the two-point function is fairly involved and is relegated to Appendix 2. The result (after symmetrising in x'' and x') can be written as

$$\begin{aligned}
&\operatorname{Tr}\langle 0|\psi(x'')\bar{\psi}(x')|0\rangle \\
&\quad = -\frac{\eta'\eta''}{96\pi^2}\sum_{s_3=s_m}^{s}(-1)^{2s_3}\frac{\Gamma(\frac{1}{2}+\nu+s_3)\Gamma(\frac{1}{2}-\nu+s_3)}{\Gamma(1+2s_3)(1+\delta_{s_3,0})}\frac{1}{\Delta x}\frac{\mathrm{d}}{\mathrm{d}\Delta x} \\
&\quad\quad \times\left\{\frac{1}{2}\left[\left(\frac{2\eta''-\Delta n+\Delta x}{2(\eta'\eta'')^{1/2}}\right)^{2s_3}+\left(\frac{2\eta'+\Delta n-\Delta x}{2(\eta'\eta'')^{1/2}}\right)^{2s_3}+(\Delta x\leftrightarrow-\Delta x)\right]\right. \\
&\quad\quad \left.\times F\left(\frac{1}{2}+\nu+s_3,\frac{1}{2}-\nu+s_3;1+2s_3;1+\frac{\Delta\eta^2-\Delta x^2}{4\eta'\eta''}\right)\right\}
\end{aligned} \tag{4.1}$$

where $\Delta\eta=\eta''-\eta'$, $\Delta x=|\boldsymbol{x}''-\boldsymbol{x}''|$ and

$$s_m=\begin{cases}0 & \text{integral spin}\\ \frac{1}{2} & \text{half-odd-integral spin.}\end{cases}$$

It is easily seen that in the case $s=0$ this reduces to the result obtained by previous authors (Bunch and Davies 1978, Candelas and Raine 1975, Dowker and Critchley 1976).

Since we only require the finite part of (4.1) we can immediately set $\Delta\eta=0$ and expand in terms of Δx, using for example equation 15.3.10 of Abramowitz and Stegun (1964).† The only finite term which will contribute a factor R^2/m^2 (and thus contribute to the anomaly) when expanded as an asymptotic series in R/m is

$$(R/96\pi^2)\sum_{s_3=s_m}^{s}(-1)^{2s_3}(1+\delta_{s_3,0})^{-1}(\tfrac{1}{4}+3s_3^2-\nu^2)[\psi(\tfrac{1}{2}+\nu+s_3)+\psi(\tfrac{1}{2}-\nu+s_3)]. \tag{4.2}$$

With ν for a conformally invariant scalar field being given by (3.14), the ψ functions in (4.2) can be expanded in terms of R/m^2 (Abramowitz and Stegun 1964, equation (6.3.18) giving the $1/m^2$ term of (4.2) as

$$\begin{aligned}
&\frac{R^2}{12m^2}\frac{(-1)^{2s}}{96\pi^2}\sum_{s_3=s_m}^{s}\frac{1}{1+\delta_{s_3,0}}\left\{\frac{1}{2}\left(\frac{2s-1}{2}\right)^4-\frac{1}{4}\left(\frac{2s-1}{2}\right)^2\right. \\
&\qquad\left.-\frac{17}{480}+s_3^2\left[\frac{1}{4}-3\left(\frac{2s-1}{2}\right)^2\right]+\frac{5}{2}s_3^4\right\}.
\end{aligned} \tag{4.3}$$

Performing the sum in (4.3) and recalling that it is m^2 multiplied by (4.3) which when *subtracted* from (2.19) gives the anomaly, we find

$$\langle T^\mu_\mu\rangle_{\text{anomalous}}=[(-1)^{2s}(2s+1)/2880\pi^2][1-5s(5s-1)]R^2/12.$$

Noting that in de Sitter space $R_{\mu\nu}R^{\mu\nu}-\frac{1}{3}R^2=-\frac{1}{12}R^2$, and bearing in mind (1.2*a*), this finally gives for the anomaly coefficient k_2

$$k_2=[(-1)^{2s}(2s+1)/2880\pi^2][1-5s(5s-1)] \tag{4.4}$$

a remarkable simple result after such a lot of calculation.

† By finite we mean non-infinite and non-zero as the points come together.

5. Discussion of the results

Let us first turn our attention to the special case in which a comparison of (4.4) with the results of other work is possible. The only results available to us are for spin-zero (scalar) and spin-one-half (neutrino) fields. The spin-one field for which results have been obtained by Dowker and Critchley (1977) and Christensen (1978) is not the same as the spin-one Weinberg-type field considered here, as it consists of a gauge field part transforming under the $(\frac{1}{2}, \frac{1}{2})$ representation and a contribution from ghost fields which transform under the (0, 0) representation (see Christensen and Duff 1978).

For spins zero and one-half, equation (4.4) gives

$$k_2 = \frac{1}{2880\pi^2}\begin{cases} 1 & s = 0 \\ \frac{11}{2} & s = \frac{1}{2} \end{cases}$$

in agreement with previous calculations (e.g. Dowker and Critchley 1977, Bunch and Davies 1977, Christensen 1978).

Thus convinced of the validity of our results we can proceed to write down all the anomaly coefficients for Weinberg-type fields. Specialising Christensen and Duff's (1978) result to these fields gives

$$k_1 = [(-1)^{2s}(2s+1)/2880\pi^2]\{1 + s(s+1)[6s(s+1) - 7]\}. \tag{5.1}$$

Our result (4.4) is

$$k_2 = [(-1)^{2s}(2s+1)/2880\pi^2][1 - 5s(5s-1)]. \tag{5.2}$$

Combining (5.1) and (5.2) using (1.2*b*) gives

$$k_3 = [(-1)^{2s}(2s+1)/2880\pi^2]\{1 + s(s-1)[4s(s+3) + 3]\} \tag{5.3}$$

and (1.2*a*) is

$$k_4 = 0$$

giving all the anomaly coefficients appearing in (1.1).

We should point out that the status of the k_3, $\square R$, anomaly coefficient is not quite the same as the others, as such a term can be manipulated by the addition of an R^2 counterterm in the gravitational action. What is more, for fields which are not conformally invariant in n dimensions, dimensional regularisation, which is used in obtaining (1.2*b*), will give a different value of k_3 from that obtained using point splitting (Christensen 1978). The determination of the conformal properties of Weinberg fields in n dimensions is not a straightforward task due to the rotation group algebra involved, so we prefer to leave the status quo as far as k_3 is concerned. In any case there are quite strong physical grounds for the complete removal of the $\square R$ anomaly using an R^2 counterterm (Horowitz and Wald 1978).

We finally note that a knowledge of these coefficients allows one to write down the entire stress tensor for the fields in an arbitrary Robertson–Walker spacetime (Bunch and Davies 1977, Bunch 1978).

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Appendix 1

In this Appendix we consider the conformal properties of (2.16) and (2.21). We start by deriving (2.20). For this the transformation of the spinor connection (2.12) under (2.17) is needed:

$$\bar{\Gamma}_\mu = \Gamma_\mu - \Omega^{-1}\sigma^{\alpha\beta} V^\nu_\alpha V_{\beta\mu}\Omega_{,\nu}. \tag{A1.1}$$

Making use of (A1.1) and equations (2.13)–(2.15) in $\bar{\nabla}^2\psi$ (the conformal transform of the first term in (2.16)) we obtain

$$\begin{aligned}\bar{\nabla}^2\psi = \Omega^{-1} V^\mu_\alpha\{&\delta^\alpha_\gamma\delta^n_m\,\partial_\mu + \delta^\alpha_\gamma[(\Gamma_\mu)^n_m - (\sigma^{\epsilon\delta})^n_m\Omega^{-1}V^\omega_\epsilon V_{\delta\mu}\Omega_{,\omega}]\\ &+\delta^n_m[(\Gamma_\mu)^\alpha_\gamma - (V^{\alpha\omega}V_{\gamma\mu} - V^\omega_\gamma V^\alpha_\mu)\Omega_{,\omega}\Omega^{-1}]\}\\ &\times\Omega^{-1}V^\gamma_\nu\{\delta^m_p\,\partial^\nu + (\Gamma^\nu)^m_p - (\sigma^{\xi\phi})^m_p\Omega^{-1}V^\sigma_\xi V^\nu_\phi\Omega_{,\sigma}\}\psi^p.\end{aligned}$$

After quite some algebra, making use of

$$[\Gamma_\mu, \sigma^{\xi\phi}]V^\mu_\phi V^\sigma_\xi = -V^\sigma_{\beta;\mu}V^\mu_\sigma\sigma^{\beta\phi} - V^\sigma_\phi V^\mu_{\beta;\mu}\sigma^{\phi\beta} \tag{A1.2}$$

and

$$\eta_{\gamma\delta}\sigma^{\alpha\gamma}\sigma^{\beta\delta} = -\sigma^{\alpha\beta} - \eta^{\alpha\beta}s(s+1) \tag{A1.3}$$

we find

$$\begin{aligned}\bar{\nabla}^2\psi = \Omega^{-2}\nabla^2\psi &+ 2\Omega^{-3}\Omega_{,\mu}[\partial^\mu + \Gamma^\mu]\psi - \Omega^{-4}s(s+1)\Omega_{,\omega}\Omega^{,\omega}\psi\\ &-2\Omega^{-3}\Omega_{,\sigma}V^\mu_\phi V^\sigma_\xi\sigma^{\xi\phi}[\partial_\mu + \Gamma_\mu]\psi.\end{aligned} \tag{A1.4}$$

We have dropped the indicies on the fields as no confusion can now arise. Next, making the field transformation (2.18) we arrive, after some more straightforward algebra, at (2.20).

We now mention briefly how the flat space modes satisfy (2.9), as this allows us to give some of the properties of the quantities $\chi(\hat{p}, s_3)$. From Weinberg (1964*b*), the positive frequency, massless $(s, 0)$ modes are

$$\psi^+_{(s,0)}(x) = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^{3/2}}(2p)^{s-1/2}\exp(-\mathrm{i}\boldsymbol{p}\,.\,\boldsymbol{x})\chi(\hat{p}, -s)a^+(\boldsymbol{p}, -s) \tag{A1.5}$$

where $\chi_n(\hat{p}, -s)$ is given by (3.7) in terms of

$$D^{(s)}_{n,-s}[R(\hat{p})] = [\exp(-\mathrm{i}\hat{n}\,.\,\boldsymbol{J}\theta)]_{n,-s} \tag{A1.6}$$

$$\hat{n} = (-p_y, p_x, 0)/(p_x^2 + p_y^2)^{1/2}$$

$$\theta = \cos^{-1}(p_z/p)$$

which is the rotation that maps the z axis into the direction $\hat{p}$ of $\boldsymbol{p}$.

For the $(s, 0)$ fields, using (2.4) and (2.5) we can write the first term in (2.9) as an η component

$$-\boldsymbol{J}\,.\,\boldsymbol{\nabla}\psi_{(s,0)} \tag{A1.7}$$

and spatial components

$$\boldsymbol{J}+\mathrm{i}\boldsymbol{J}\times\boldsymbol{\nabla}\psi_{(s,0)}. \tag{A1.8}$$

Using the well known relation (e.g. Merzbacher 1970, equation (16.9))

$$\exp(\mathrm{i}\hat{n}\,.\,\boldsymbol{J}\theta)\boldsymbol{J}\exp(-\mathrm{i}\hat{n}\,.\,\boldsymbol{J}\theta)=\hat{n}(\hat{n}\,.\,\boldsymbol{J})-\hat{n}\times(\hat{n}\times\boldsymbol{J})\cos\theta+\hat{n}\times\boldsymbol{J}\sin\theta \tag{A1.9}$$

we can easily derive

$$\boldsymbol{p}\,.\,\boldsymbol{J}\chi(\hat{p},-s)=-sp\chi(\hat{p},-s) \tag{A1.10}$$

and

$$p\boldsymbol{J}\chi(\hat{p},-s)+\mathrm{i}\boldsymbol{p}\times\boldsymbol{J}\chi(\hat{p},-s)=-s\boldsymbol{p}\chi(\hat{p},-s) \tag{A1.11}$$

which when used in conjunction with (A1.5, 7, 8) gives (2.9). The $(0, s)$ case is similar.

To show the conformal invariance of the curved space equivalent of (2.9), namely (2.21), is quite straightforward, provided one makes use of (A1.1) and (A1.3), the procedure following lines similar to those used above to show the conformal invariance of (2.16).

Appendix 2

We wish to show the calculation of the integrals leading to (4.1). Using (3.11) and (3.12) we have

$$\tfrac{1}{2}(\mathrm{Tr}\langle\psi(x'')\bar{\psi}(x')\rangle+(x''\leftrightarrow x'))=(\eta''\eta'R/96\pi^2)\sum_{s_3=-s}^{s}I(s_3)+(x''\leftrightarrow x') \tag{A2.1}$$

where

$$I(s_3)=\frac{(-1)^{s_3}}{\Delta x}\int_0^\infty \mathrm{d}p\,\sin p\,\Delta x\,W_{s_3,\nu}(2\mathrm{i}p\eta'')W_{-s_3,\nu}(-2\mathrm{i}p\eta').$$

Writing the W Whittaker functions in terms of M Whittaker functions (Gradshteyn and Ryzhik 1965, equation 9.220(4)) we have

$$\begin{aligned}I(s_3)=&\frac{\Gamma(-2\nu)^2I_{++}(s_3)}{\Gamma(\frac{1}{2}-\nu-s_3)\Gamma(\frac{1}{2}-\nu+s_3)}+\frac{\Gamma(2\nu)^2I_{--}(s_3)}{\Gamma(\frac{1}{2}+\nu-s_3)\Gamma(\frac{1}{2}+\nu+s_3)}\\&+\frac{\Gamma(-2\nu)\Gamma(2\nu)}{\Gamma(\frac{1}{2}-\nu-s_3)\Gamma(\frac{1}{2}+\nu+s_3)}[I_{+-}(s_3)+I_{-+}(s_3)]\end{aligned} \tag{A2.2}$$

where

$$I_{\pm\pm}=\frac{(-1)^{s_3}}{\Delta x}\int_0^\infty \mathrm{d}p\,\sin p\,\Delta x\,M_{s_3,\pm\nu}(2\mathrm{i}p\eta'')M_{-s_3,\pm\nu}(-2\mathrm{i}p\eta') \tag{A2.3}$$

and $I_{\pm\mp}$ simply has $\mp\nu$ in the second Whittaker function.

$I_{\pm\mp}$ can be evaluated (Gradshteyn and Ryzhik 1965, equation 7.622(3)) in terms of a generalised hypergeometric function of two variables as

$$\begin{aligned}I_{\pm\mp}=&\frac{(-1)^{s_3}}{2\mathrm{i}\,\Delta x}\{(2\mathrm{i}\eta')^{\pm\nu+1/2}(-2\mathrm{i}\eta'')^{\mp\nu+1/2}[\mathrm{i}(\Delta\eta-\Delta x)]^{-2}\\&\times F_2(2;\tfrac{1}{2}\pm\nu-s_3,\tfrac{1}{2}\mp\nu+s_3;1\pm2\nu;2\eta''/(\Delta\eta-\Delta x),-2\eta'/(\Delta\eta-\Delta x))\\&-(\Delta x\to-\Delta x)\}.\end{aligned} \tag{A2.4}$$

Using the symmetries of F_2, it is easily shown that the combination $I_{+-}(s_3)+I_{-+}(-s_3, \eta'\leftrightarrow\eta'')$ (recall $\Delta x \equiv |\boldsymbol{x}''-\boldsymbol{x}'|$), which appears in (A2.1), is zero. Thus there is no contribution from the $I_{\pm\mp}$ terms to (A2.1).

To calculate $I_{\pm\pm}(s_3)$ we write it as

$$I_{\pm\pm}(s_3) = (1/\Delta x)(\mathrm{d}/\mathrm{d}\Delta x) I^{(1)}_{\pm\pm}(s_3) \tag{A2.5}$$

where

$$I^{(1)}_{\pm\pm}(s_3) = -\int_0^\infty \mathrm{d}p\, p^{-1} \cos p\,\Delta x \exp(\mathrm{i}\pi s_3) M_{s_3,\pm\nu}(2\mathrm{i}p\eta'') M_{-s_3,\pm\nu}(-2\mathrm{i}p\eta').$$

Using Erdélyi (1953, equation 6.9(7)) and then Gradshteyn and Ryzhik (1965, equation 7.622(3)) one has

$$\begin{aligned} I^{(1)}_{\pm}(s_3) + I^{(1)}_{\pm}(-s_3) &= -\mathrm{Re}\{(4\eta'\eta'')^{\pm\nu+1/2} \exp(\mathrm{i}\pi s_3)[\mathrm{i}(\Delta\eta-\Delta x)]^{\mp 2\nu-1}\Gamma(1\pm 2\nu) \\ &\quad \times F_2(1\pm 2\nu; \tfrac{1}{2}\pm\nu-s_3, \tfrac{1}{2}\pm\nu+s_3; 1\pm 2\nu; 1\pm 2\nu; 2\eta''/(\Delta\eta-\Delta x), \\ &\quad -2\eta'/(\Delta\eta-\Delta x)) + (s_3\to -s_3)\} \\ &= \sin\pi(\pm\nu-s_3)\left(\frac{\Delta\eta-\Delta x-2\eta''}{\Delta\eta-\Delta x+2\eta'}\right)^{s_3}\left(\frac{4\eta'\eta''+\Delta\eta^2-\Delta x^2}{-4\eta'\eta''}\right)^{-1/2\mp\nu} \\ &\quad \times\Gamma(1\pm 2\nu) F(\tfrac{1}{2}\pm\nu-s_3, \tfrac{1}{2}\pm\nu+s_3; 1\pm 2\nu; 4\eta'\eta''/(4\eta'\eta''+\Delta\eta^2-\Delta x^2)) \\ &\quad + (s_3\to -s_3) \end{aligned} \tag{A2.6}$$

where we have converted to an ordinary hypergeometric function using Erdélyi (1953, equation 5.10(3)).

Substituting this into (A2.5) which is then used in (A2.2) and making use of the identity

$$\begin{aligned} &\Gamma(\mp 2\nu)\Gamma(1\pm 2\nu)\Gamma(\tfrac{1}{2}\mp\nu-s_3)\sin\pi(\pm\nu-s_3)/\Gamma(\tfrac{1}{2}\mp\nu+s_3) \\ &\quad = (-1)^{2s_3+1}\Gamma(\tfrac{1}{2}+\nu-s_3)\Gamma(\tfrac{1}{2}-\nu-s_3) \end{aligned} \tag{A2.7}$$

we obtain

$$\begin{aligned} I(s_3)+I(-s_3) &= \frac{\Gamma(\frac{1}{2}+\nu-s_3)\Gamma(\frac{1}{2}-\nu-s_3)}{\Gamma(1-2s_3)}\,\frac{(-1)^{2s_3+1}}{2\Delta x}\,\frac{\mathrm{d}}{\mathrm{d}\Delta x} \\ &\quad \times\left\{\left(\frac{2\eta''+\Delta x-\Delta\eta}{2\eta'-\Delta x+\Delta\eta}\right)^{s_3}\left(\frac{4\eta'\eta''+\Delta\eta^2-\Delta x^2}{4\eta'\eta''}\right)^{-s_3}\left[\frac{\Gamma(-2\nu)\Gamma(1-2s_3)}{\Gamma(\frac{1}{2}-\nu-s_3)^2}\right.\right. \\ &\quad \times\left(\frac{4\eta'\eta''+\Delta\eta^2-\Delta x^2}{4\eta'\eta''}\right)^{-1/2-\nu+s_3} F\left(\frac{1}{2}+\nu-s_3, \frac{1}{2}+\nu+s_3;\right. \\ &\quad \left.\left.\left. 1+2\nu; \frac{4\eta'\eta''}{4\eta'\eta''+\Delta\eta^2-\Delta x^2}\right) + (\nu\to-\nu)\right]\right\} + (s_3\to -s_3). \end{aligned} \tag{A2.8}$$

The term inside square brackets in (A2.8) is recognisable as the RHS of Abramowitz and

Stegun (1964, equation (15.3.7)) with $a=\frac{1}{2}+\nu-s_3$, $b=\frac{1}{2}-\nu-s_3$, $c=1-2s_3$, giving

$$I(s_3)+I(-s_3)=(-1)^{2s_3+1}\frac{\Gamma(\frac{1}{2}+\nu-s_3)\Gamma(\frac{1}{2}-\nu-s_3)}{\Gamma(1-2s_3)}\frac{1}{2\Delta x}\frac{\mathrm{d}}{\mathrm{d}\Delta x}$$
$$\times\left[\left(\frac{2\eta''-\Delta\eta+\Delta x}{2\eta'+\Delta\eta-\Delta x}\right)^{s_3}\left(\frac{4\eta'\eta''+\Delta\eta^2-\Delta x^2}{4\eta'\eta''}\right)^{-s_3}F\left(\frac{1}{2}+\nu-s_3,\frac{1}{2}-\nu-s_3;\right.\right.$$
$$\left.\left.1-2s_3;1+\frac{\Delta\eta^2-\Delta x^2}{4\eta'\eta''}\right)\right]+(s_3\to -s_3). \qquad \text{(A2.9)}$$

Now the hypergeometric function in (A2.9) by itself is not defined for $s_3>0$ ($s_3=\frac{1}{2},1,\frac{3}{2},\ldots$), but the limit of the hypergeometric function divided by $\Gamma(1-2s_3)$ is, and is given by (Abramowitz and Stegun 1964, equation 15.1.2)

$$\frac{(\frac{1}{2}+\nu-s_3)_{2s_3}(\frac{1}{2}-\nu-s_3)_{2s_3}}{\Gamma(1+2s_3)}\left(\frac{4\eta'\eta''+\Delta\eta^2-\Delta x^2}{4\eta'\eta''}\right)^{2s_3}$$
$$\times F(\tfrac{1}{2}+\nu+s_3,\tfrac{1}{2}-\nu+s_3;1+2s_3;1+(\Delta\eta^2-\Delta x^2)/4\eta'\eta'').$$

Thus we finally have

$$I(s_3)+I(-s_3)=(-1)^{2s_3+1}\frac{\Gamma(\frac{1}{2}+\nu+s_3)\Gamma(\frac{1}{2}-\nu+s_3)}{\Gamma(1+2s_3)}\frac{1}{2\Delta x}\frac{\mathrm{d}}{\mathrm{d}\Delta x}$$
$$\times\left\{\left[\left(\frac{2\eta''+\Delta\eta+\Delta x}{2(\eta'\eta'')^{1/2}}\right)^{2s_3}+\left(\frac{2\eta'+\Delta\eta-\Delta x}{2(\eta'\eta'')^{1/2}}\right)^{2s_3}\right]\right.$$
$$\left.\times F\left(\frac{1}{2}+\nu+s_3,\frac{1}{2}-\nu+s_3;1+2s_3;1+\frac{\Delta\eta^2-\Delta x^2}{4\eta'\eta''}\right)\right\}, \qquad \text{(A2.10)}$$

which when substituted in (A2.1) gives (4.1).

We finally note that, because we only require the part of (A2.1) which is finite and non-zero when the points come together, we have not included parallel propagators in the calculation.

References

Abramowitz M and Stegun I A 1964 *Handbook of Mathematical Tables* (New York: Dover)
Birrell N D 1978 *Proc. R. Soc.* **361** 513
Birrell N D and Davies P C W 1978 *Nature* **272** 35
Brown L S 1977 *Phys. Rev.* D **15** 1469
Brown L S and Cassidy J P 1977 *Phys. Rev.* D **15** 2810
Brown M R and Dutton C R 1978 *University of Texas at Austin, Preprint*
Bucholz H 1969 *The Confluent Hypergeometric Function* (Berlin: Springer)
Bunch T S 1978 *Phys. Rev.* D **15** 1844
Bunch T S, Christensen S M and Fulling S A 1978 *Harvard University, Centre for Astrophysics, Preprint 906*
Bunch T S and Davies P C W 1977 *Proc. R. Soc.* A **356** 569
—— 1978 *Proc. R. Soc.* A **360** 117
Candelas P and Raine D J 1975 *Phys. Rev.* D **12** 965
Capper D M and Duff M J 1974 *Nuovo Cim.* A **23** 173
Christensen S M 1978 *Phys. Rev.* D **17** 946
Christensen S M and Duff M J 1978 *Phys. Lett.* B **76** 571
Christensen S M and Fulling S A 1977 *Phys. Rev.* D **15** 2088
Davies P C W, Fulling S A, Christensen S M and Bunch T S 1977 *Ann. Phys.* **109** 108

Davies P C W, Fulling S A and Unruh W G 1976 *Phys. Rev.* D **13** 2720
Deser S, Duff M J and Isham C J 1976 *Nucl. Phys.* B **111** 45
DeWitt B S 1965 *The Dynamical Theory of Groups and Fields* (New York: Gordon and Breach)
—— 1975 *Phys. Rep.* C **19** 295
Dowker J S 1967 *Proc. R. Soc.* A **297** 351
—— 1977 *Manchester University, Preprint*
—— 1978 *J. Phys. A: Math. Gen.* **11** 347
Dowker J S and Critchley R 1976 *Phys. Rev.* D **12** 3224
—— 1977 *Phys. Rev.* D **16** 3390
Dowker J S and Dowker Y P 1966a *Proc. Phys.* Soc. **87** 65
—— 1966b *Proc. R. Soc.* A **294** 175
Duff M J 1977 *Nucl. Phys.* B **125** 334
Duncan A 1977 *Phys. Lett.* B **66** 170
Erdélyi A 1953 *Higher Transcendental Functions* Vol. I (New York: McGraw-Hill)
Gradshteyn I S and Ryzhik I W 1965 *Tables of Integrals, Series and Products* (New York: Academic)
Grensing G 1977 *J. Phys. A: Math. Gen.* **10** 1687
Horowitz G T and Wald M W 1978 *Phys. Rev.* D **17** 414
Kazarinoff N D 1955 *Trans. Am. Math. Soc.* **78** 305
Merzbacher E 1970 *Quantum Mechanics* (New York: Wiley)
Mohan G 1968 *Higher Spin Fields* in *Lectures in Theoretical Physics* Vol. XB ed. A O Barut and W E Brittin (New York: Gordon and Breach)
Nachtmann O 1967 *Commun. Math. Phys.* **6** 1
Parker L and Fulling S A 1974 *Phys. Rev.* **9** 341
Schwinger J 1951 *Phys. Rev.* **82** 664
Tsao H S 1977 *Phys. Lett.* B **68** 79
Weinberg S 1964a *Phys. Rev.* **133** B1318
—— 1964b *Phys. Rev.* **134** B882
—— 1964c *Quantum Theory of Massless Particles* in *Particles and Field Theory* (Brandeis Summer School)
—— 1972 *Gravitation and Cosmology* (New York: Wiley)
Whittaker E T and Watson G N 1927 *A Course of Modern Analysis* (London: CUP)
Wichmann E H 1962 *Selected Topics in the Theory of Particles and Fields* (Copenhagen: Nordita)